We'll see some applications of the gradient estimates for the scalar curvature.

Thm Let $(M^3, g(t))$ be a RF w/ $Ric(g(0)) > 0$, defined for $t \in [0,T)$. Then $\exists$ constants $C, \gamma > 0$ depending only on $g(0)$ s.t.

$$\frac{R_{min}}{R_{max}} \geq 1 - C R_{max}^{-\gamma}.$$

$\therefore$ $\frac{R_{min}}{R_{max}} \to 1$ as $t \to T$ b/c $R_{max} \to \infty$ as $t \to T$ and hence $R \to \infty$ uniformly as $t \to T$.

Proof:- We know that $R_{max} \to \infty$ as $t \to T$. By the gradient estimate on R we know that

$$|\nabla R| \leq A R_{max}^{\frac{3}{2} - \alpha} \quad \text{as } t \to T.$$

$\therefore$ for t close to T, let γ be a minimizing geodesic connecting $x, y \in M$. Then

$$|R(x) - R(y)| \leq \int_\gamma |\nabla R| \, ds \leq A R_{max}^{\frac{3}{2} - \alpha} d(x,y).$$

$\because$ M is compact $\Rightarrow$ $\exists$ (x, t) s.t. $R_{max}(t) = R(x,t)$.

Define $L(t) = \dfrac{1}{\varepsilon\sqrt{R_{max}(t)}}$ [our intuition here is that the manifold is converging to a round point].

w/ $\varepsilon > 0$. and consider $B(x(t), L(t))$.

$$\therefore \quad R(y) \geq R(x) - A R_{max}^{\frac{3}{2}-\alpha} L \geq R_{max}\left(1 - \frac{A}{\varepsilon} R_{max}^{-\alpha}\right).$$

$\because$ $R_{max} \to \infty$ as $t \to T \Rightarrow$ given $\delta > 0$ for t sufficiently close to T, we have

$$R(y) \geq (1-\delta) R_{max} \quad \forall\, y \in B(x(t), L(t)).$$

We'd prove that $B(x(t), L(t))$ is all of M for t sufficiently close to T. If we denote $N = B(x(t), L(t))$ then $\because$ we know that if

$$Rc = \frac{1}{2}\begin{bmatrix} \mu+\nu & & \\ & \lambda+\nu & \\ & & \lambda+\mu \end{bmatrix}$$

and $\lambda \leq C(\mu+\nu) \Rightarrow$

$$Rc \geq \frac{\mu+\nu}{2} g \geq \frac{\lambda}{2C} g = \frac{3\lambda}{6C} g \geq \frac{\lambda+\mu+\nu}{6C} g$$

$$= \frac{R}{6C} g = 2\beta^2 R g$$

$$\therefore \quad Ric \geq 2\beta^2 (1-\delta) R_{max}\, g$$

for any $\delta > 0$ in N for t close enough to T.

$\Rightarrow$ By Myers' Thm

$$\operatorname{diam}(N) \leq \frac{\pi}{\beta\sqrt{1-\delta}\, R_{max}} < \frac{1}{\delta\sqrt{R_{max}}} < \frac{1}{\varepsilon\sqrt{R_{max}}} = L.$$

for δ sufficiently small.

But $N = B(x(t), L(t))$ and has diameter $< L \Rightarrow$ all points of M must lie in $N \Rightarrow N$ must be all of $M \Rightarrow$

$$R(y) \geq R_{max}\left(1 - \frac{A}{\varepsilon} R_{max}^{-\alpha}\right)$$ and we get the result. $\boxtimes$

<u>Corollary</u> Let $\lambda(x,t) \geq \mu(x,t) \geq \nu(x,t)$ be the eigenvalues of the curvature operator at (x,t). Then for any $\varepsilon \in (0,1)$ $\exists$ $T_\varepsilon \in [0,T)$ s.t.

$$\min_{x \in M^3} \nu(x,t) \geq (1-\varepsilon) \max_{y \in M^3} \lambda(y,t) > 0 \quad \forall\, t \in [T_\varepsilon, T).$$

$\therefore$ the sectional curvatures are eventually approaching eachother.

<u>Proof</u> [Exercise].

we know that $\exists\ C>0$ and $\delta<1$ s.t.

$$\lambda-\nu \le C(\lambda+\mu+\nu)^{1-\delta} \Rightarrow$$

$$\nu \ge \lambda - C(\lambda+\mu+\nu)^{1-\delta} \quad \forall\ x \in M^n.$$

$$\therefore \frac{\nu}{\lambda} \ge 1 - \frac{C(\lambda+\mu+\nu)^{1-\delta}}{\lambda} \ge 1-3CR^{-\delta}.$$

Let $x, y \in M^3$ and $\eta > 0$ be given. For $T_\eta \in [0,T)$, let $T_\eta \le t < T$, we get

$$\nu(x,t) \ge (1-\eta)\lambda(x,t) \quad [\text{choose } \eta > 3CR^{-\delta}]$$

$$\ge \frac{1-\eta}{3} R(x,t)$$

$$\ge \frac{(1-\eta)^2}{3} R(y,t) \quad \left(\text{as } \frac{R_{max}}{R_{min}} \to 1\right)$$

as $1-\eta<1$)

$$\ge \frac{(1-\eta)^2}{3}(\lambda+\mu+\nu)(y,t)$$

$$\geq \frac{(1-\eta)^2}{3}\left(\lambda(y,t) + 2\nu(y,t)\right)$$

$$\geq \frac{(1-\eta)^2}{3}\left(\lambda(y,t) + 2(1-\eta)\lambda(y,t)\right)$$

(using the estimate which we started with)

$$\geq (1-\eta)^3 \lambda(y,t)$$

from which we get the result

$\therefore$ we know now that the RF on M^3 w/ $Ric(g(0)) > 0$ becomes singular in finite time T, $R \to \infty$ as $t \to T$ and the sectional curvatures get pinched together as the curvature explodes.

$\because$ in the limit as $t \to T$ we want a metric of constant sectional curvature we'd like to analyze the limiting manifold, but as $t \to T$

$M^3 \longrightarrow \bullet$ $\therefore$ we'd like to rescale the metric so that the vol^n remains constant.

$\therefore$ we look at the NRF

$$\partial_t \tilde{g} = -2\widetilde{Ric} + \frac{2\tilde{r}}{n}\tilde{g} \quad \text{where } \tilde{r} = \frac{\int \tilde{R}\, d\mu}{\int d\mu}.$$

We recall the Bishop–Günther volume comparison theorem.

Let $V_n^K(r)$ be the volume of ball of radius r in the complete, simply-connected n-dimensional space of constant sectional curvature K (which is either S_R^n, $\mathbb{R}^n$ or $\mathbb{H}_R^n$). Suppose (M^n, g) is Riemannian and $p \in M$. Then

1) If $\exists$ a constant $a > 0$ s.t. $Ric \geq (n-1)ag$ then
$$Vol(B(p,r)) \leq V_n^a(r).$$

2) If $\exists$ a constant b s.t. all sectional curvatures of (M^n, g) are <u>bounded above</u> by b and the exponential map is injective on $B(p,r)$, then
$$Vol(B(p,r)) \geq V_n^b(r).$$